

Topological waves in magnetized cold plasma

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Theory seminar

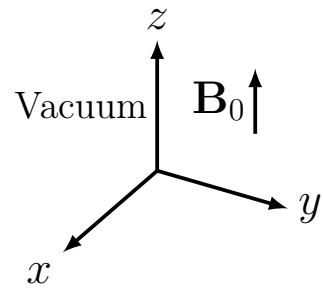
July 15, 2021

Outline

- Prelude: waves in plasmas
- Topological properties of waves
- Topological plasma waves
- Conclusions and discussions

- **Prelude: waves in plasmas**
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Bulk waves: EM waves in vacuum

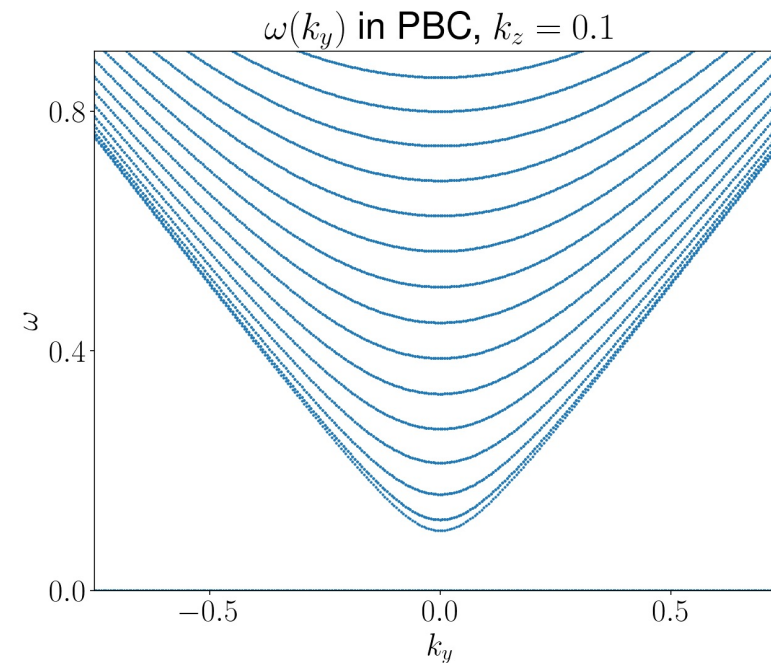
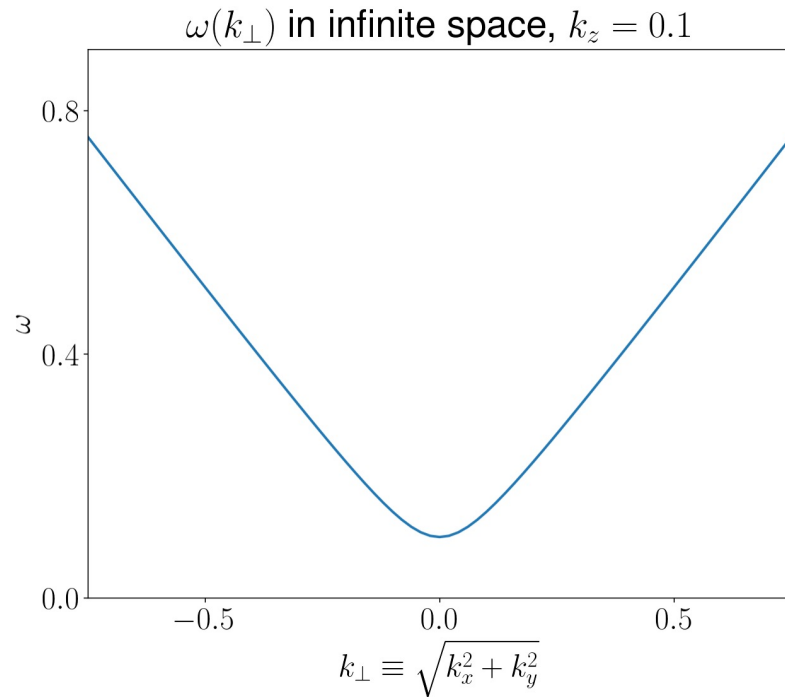


- In an infinitely large space:

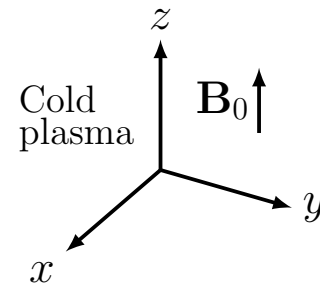
$$\omega^2 = c^2 (k_{\perp}^2 + k_z^2)$$

- With PBC: $f(x + L) = f(x)$

$$\omega^2 = c^2 (k_x^2 + k_y^2 + k_z^2), \quad k_x = \frac{2\pi n}{L}, \quad n \in \mathbb{Z}$$

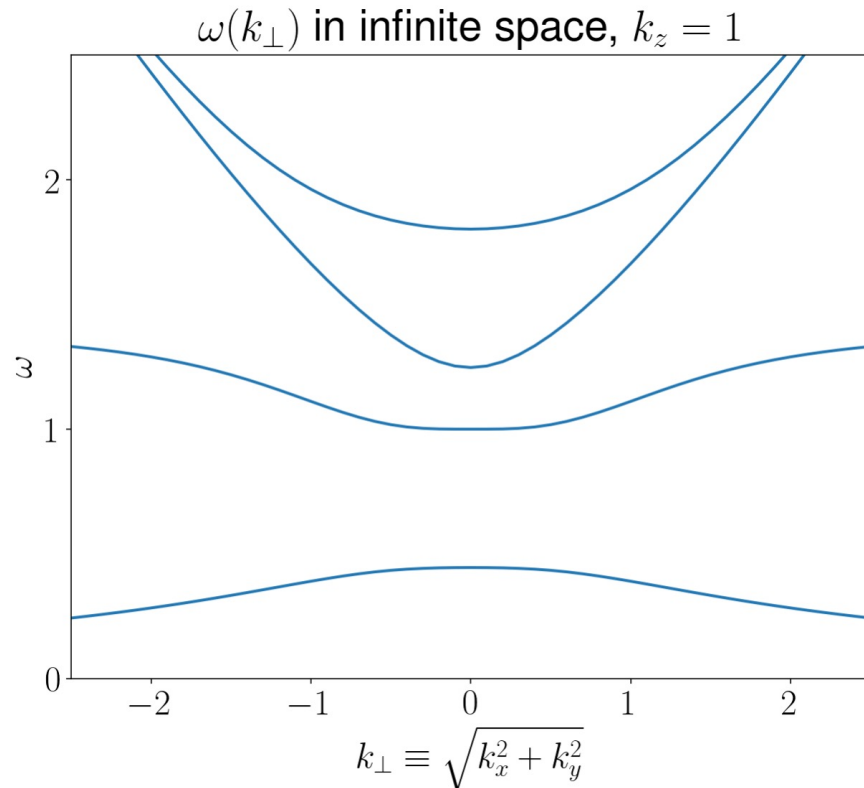


Bulk waves: waves in cold plasmas



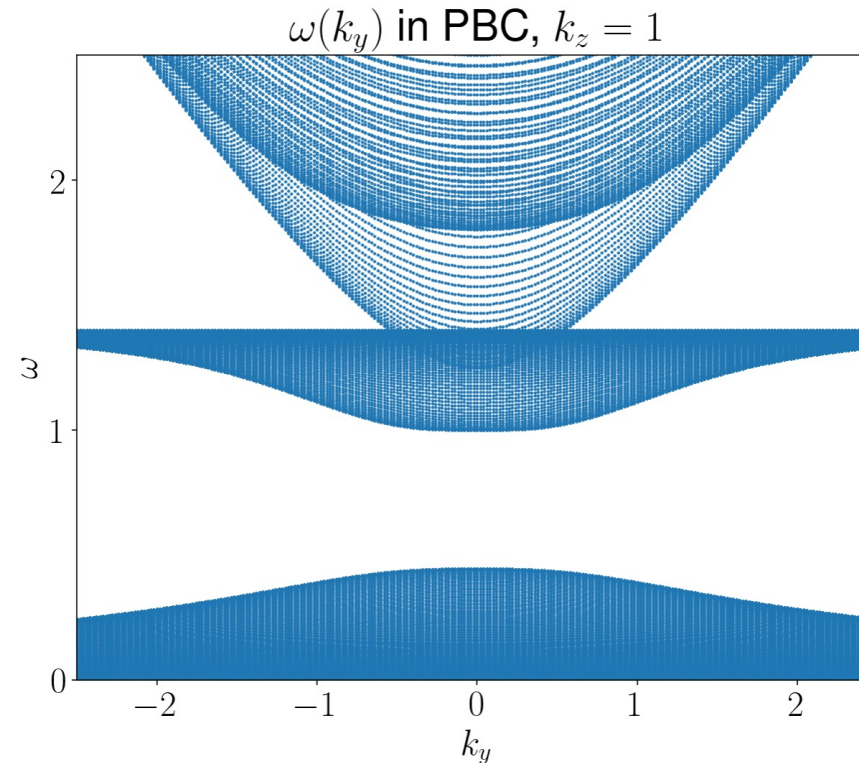
- In an infinitely large space:

$$\omega = \omega(k_{\perp}, k_z)$$



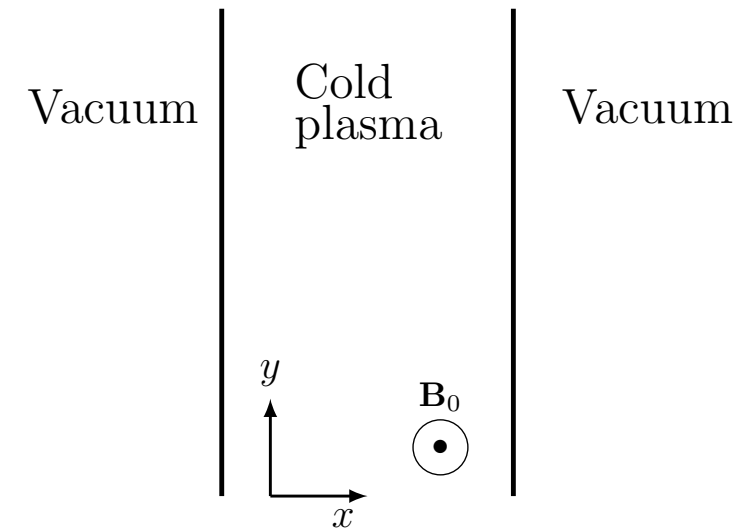
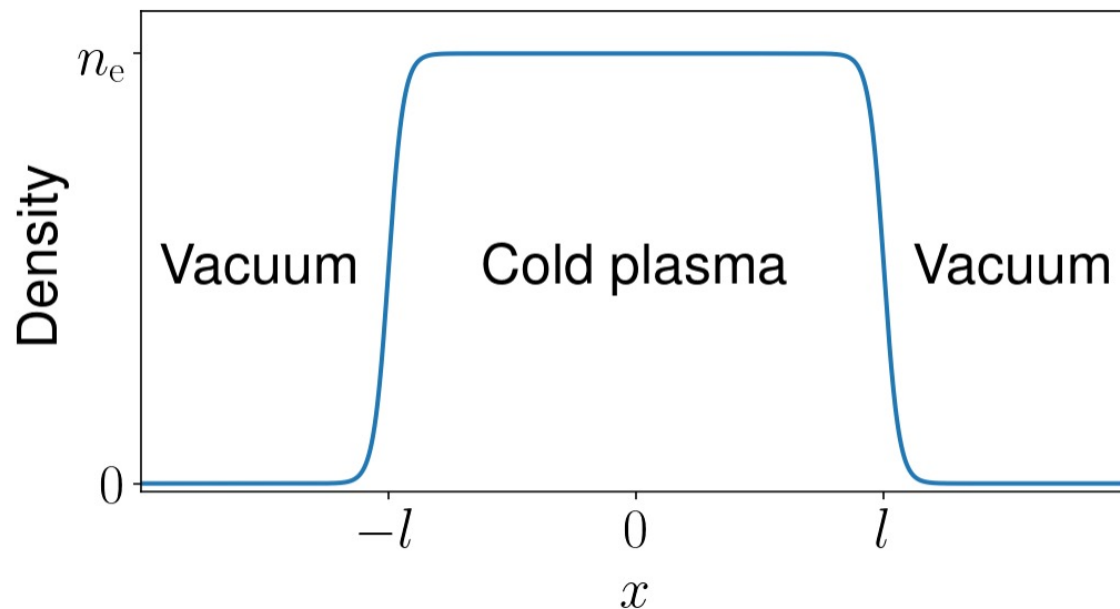
- With PBC: $f(x + L) = f(x)$

$$\omega = \omega\left(\sqrt{k_x^2 + k_y^2}, k_z\right), \quad k_x = \frac{2\pi n}{L}, \quad n \in \mathbb{Z}$$



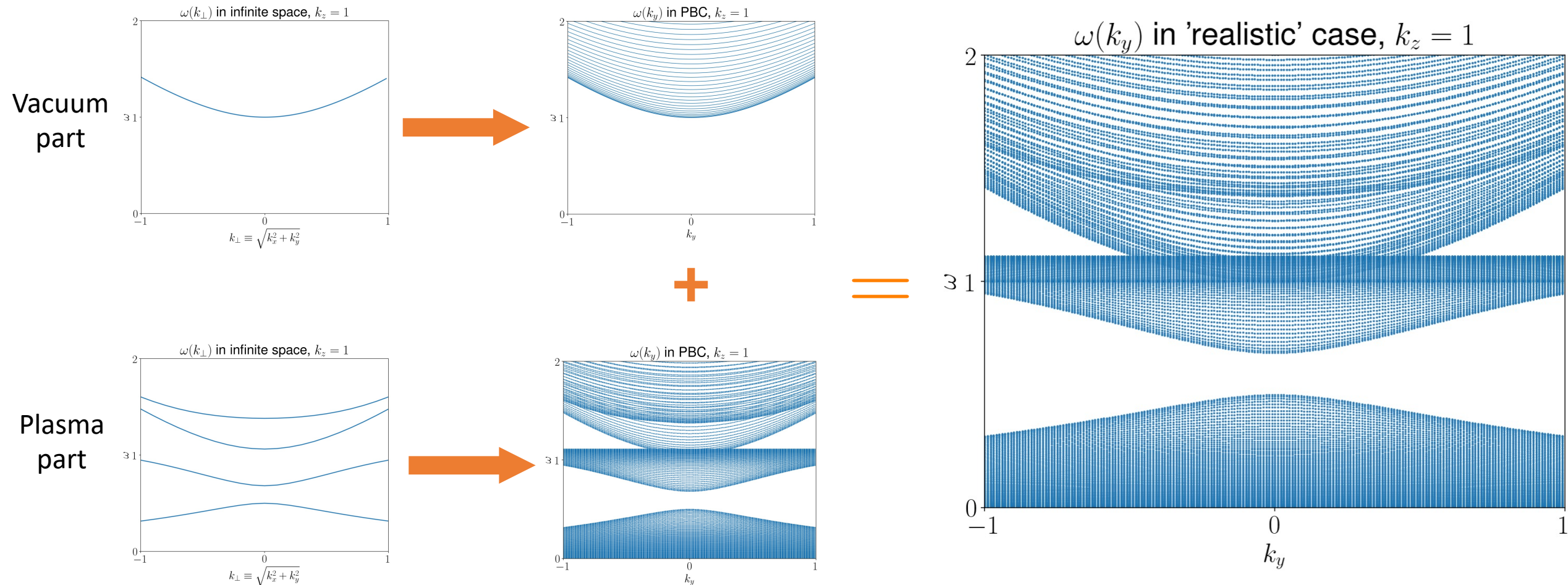
Systems with both plasma and vacuum

- **Question:** when we have a 'realistic' boundary between plasma and vacuum, what spectrum shall we expect?
- Assume the plasma is non-uniform in x direction, and infinite in other directions.



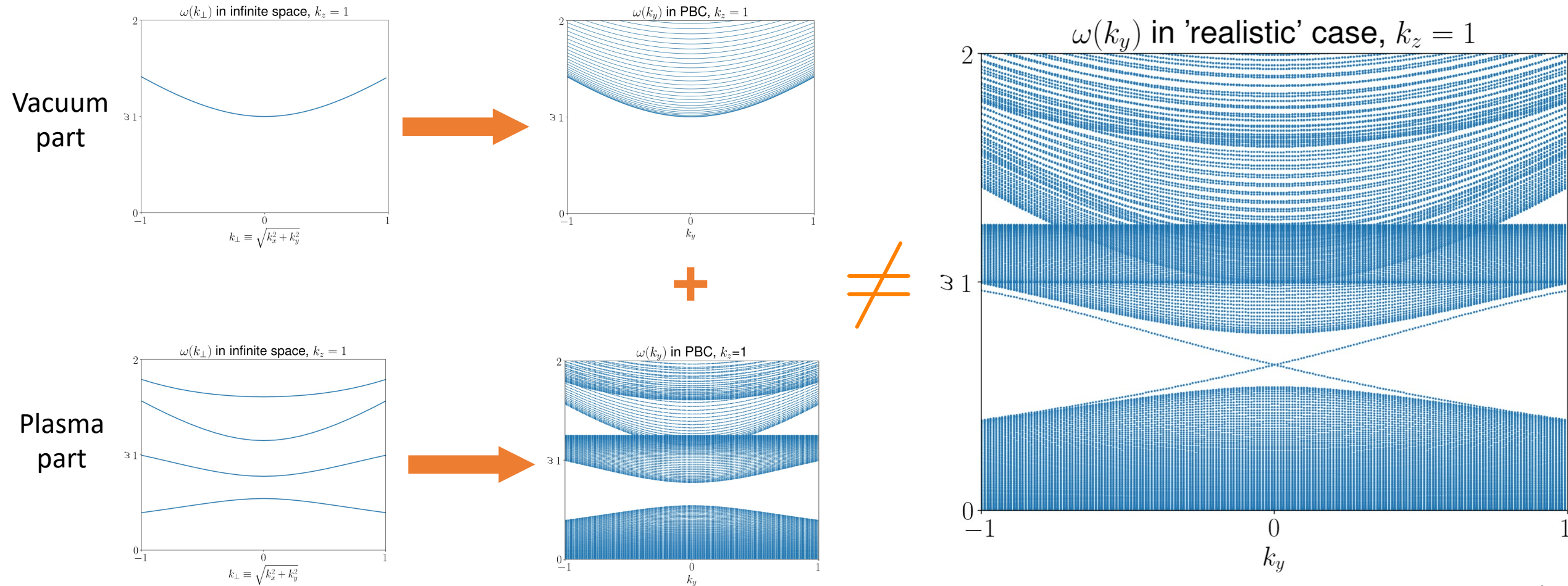
Spectrum in 'low' density case

Let $k_z = B_0 = m_e = q = c = 1$, and we take $n_e = 0.25$.



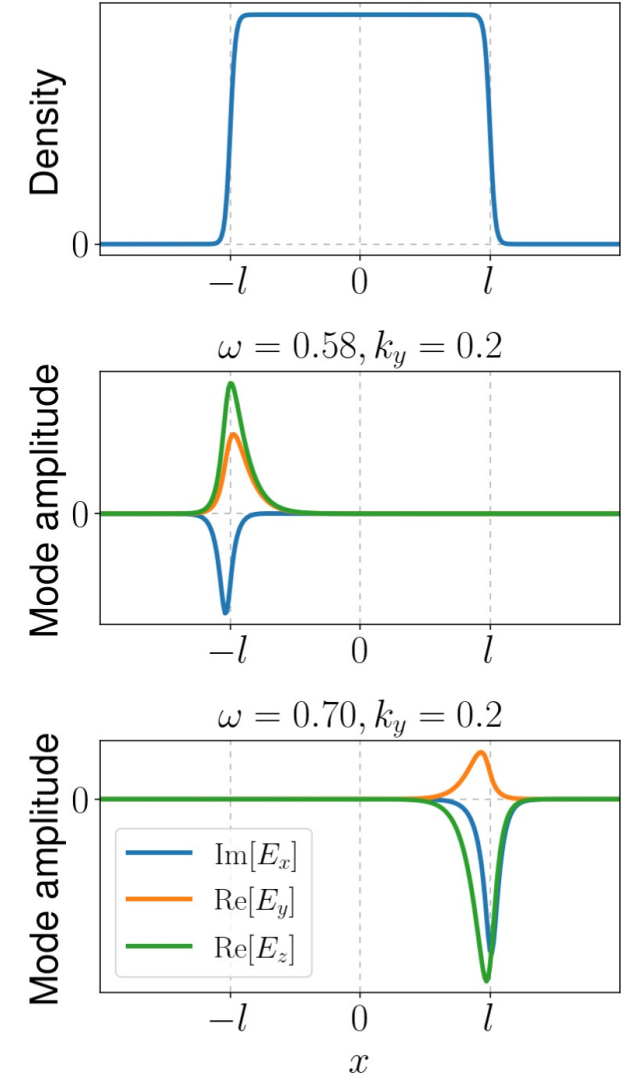
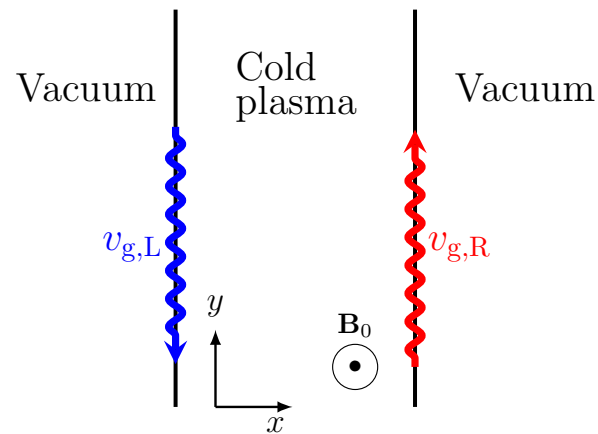
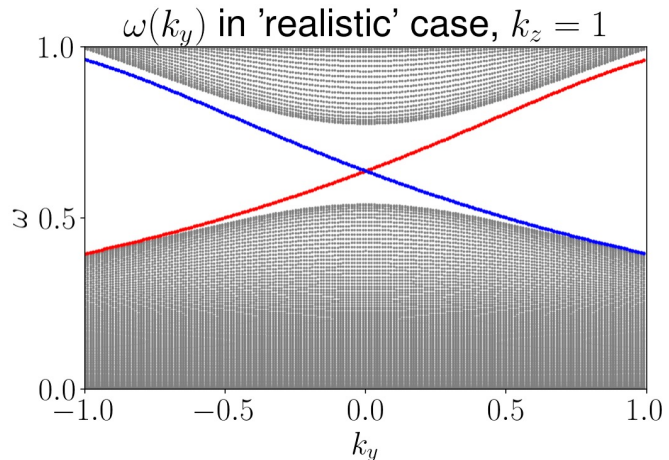
Spectrum in 'high' density case

Let $k_z = B_0 = m_e = q = c = 1$, but now we take $n_e = 0.6$.



Details of these modes

- These waves are localized at plasma-vacuum interfaces. Each interface has one edge wave.
- The group velocities of the waves at each interface are opposite. This wave is the so-called chiral edge mode.
- The existence of edge modes does not rely on the length scale of density decay.



How to understand this phenomenon?

- For given k_z and $\Omega = qB_0/m_e$, the critical density is given by:

$$\omega_{p,c} = \frac{|\Omega|}{2} \left[\sqrt{\left(\frac{ck_z}{\Omega}\right)^4 + 4 \left(\frac{ck_z}{\Omega}\right)^2} - \left(\frac{ck_z}{\Omega}\right)^2 \right]$$

- **Traditional methods:** solve the waves on both sides and match them with boundary conditions.
- **New methods:** study the ‘topology’ of the waves on each side.

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Topology of eigenvectors

- Assume we have a **uniform** system in an **infinite** space. After Fourier transform, we get an N-dimensional Hermitian eigenvalue problem in momentum space: $\mathcal{H}(\mathbf{k}) |\psi_n(\mathbf{k})\rangle = E_n(\mathbf{k}) |\psi_n(\mathbf{k})\rangle$
- Let $\mathbf{k} \in \mathcal{M}_1, |\psi_n\rangle \in \mathcal{M}_2 \subset \{z \in \mathbb{C}^N \mid \|z\| = 1\}$, then each eigenvector defines a map: $|\psi_n\rangle : \mathcal{M}_1 \rightarrow \mathcal{M}_2$.
- Now, we can discuss the topological equivalent classes of the maps, characterized by various topological invariances.
- Let \mathcal{M}_1 be 2D, define $P_n(\mathbf{k}) := |\psi_n\rangle\langle\psi_n|$, the (first) **Chern number** can be defined as:

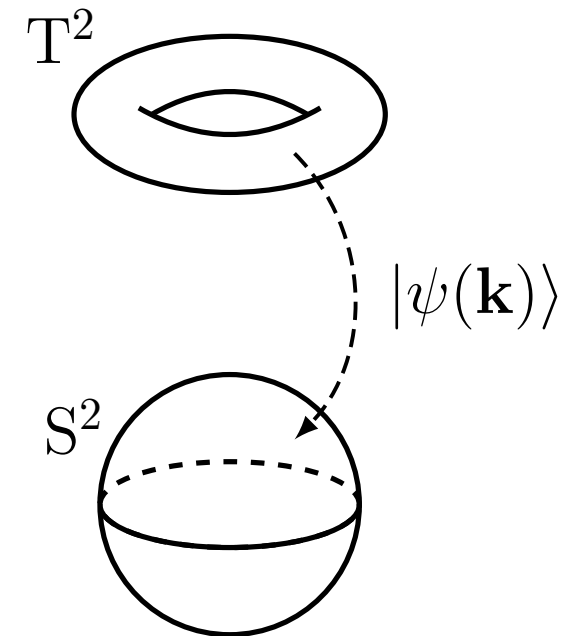
$$\begin{aligned} C_n &= -\frac{i}{2\pi} \int_{\mathcal{M}_1} \text{tr} (P_n \, dP_n \wedge dP_n) \\ &= -\frac{i}{2\pi} \int_{\mathcal{M}_1} (\langle \partial_{k_x} \psi_n | \partial_{k_y} \psi_n \rangle - \langle \partial_{k_y} \psi_n | \partial_{k_x} \psi_n \rangle) d^2\mathbf{k} \quad (\text{TKNN}) \end{aligned}$$

Example: two-level system

- Let the momentum space be 2D and periodic in both directions, i.e., the 2D Brillouin zone, $\mathbf{k} \in T^2$.
- The Hamiltonian is: $\mathcal{H}(\mathbf{k}) = \mathbf{d}(\mathbf{k}) \cdot \boldsymbol{\sigma}$, $\mathbf{d} = (d_x, d_y, d_z)$, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.
Its eigen vectors can be mapped to the Bloch sphere S^2 , e. g.,

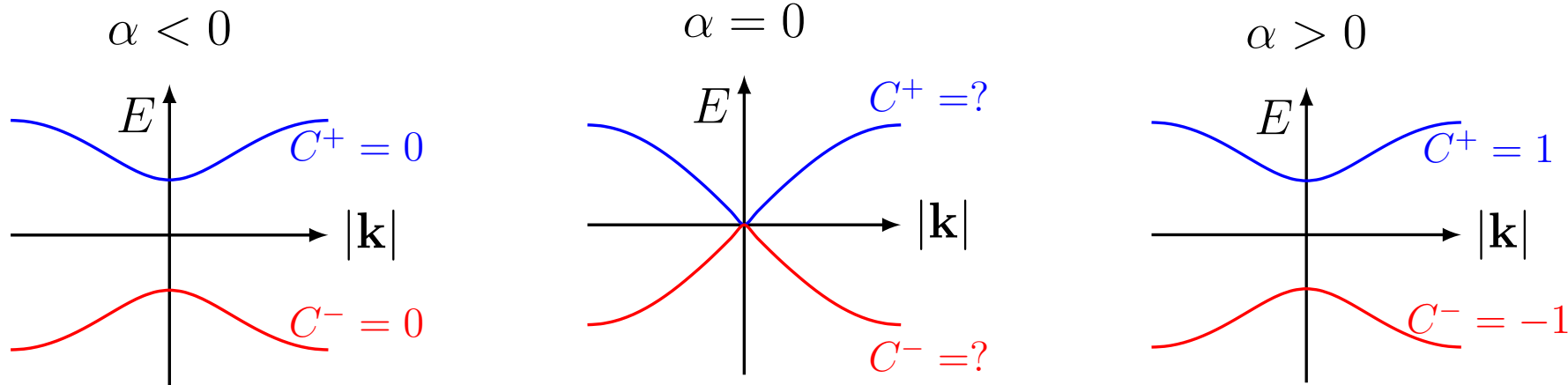
$$|\psi^+\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) e^{-i\varphi} \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix}, \quad \theta(\mathbf{k}) \in [0, \pi], \quad \varphi(\mathbf{k}) \in [0, 2\pi).$$

- Thus, $|\psi(\mathbf{k})\rangle$ defines a map $T^2 \rightarrow S^2$, which is classified by $\pi_2(S^2) = \mathbb{Z}$.
- The Chern number is: $C^+ = \frac{1}{4\pi} \int_{T^2} \hat{\mathbf{d}} \cdot \left(\partial_{k_x} \hat{\mathbf{d}} \times \partial_{k_y} \hat{\mathbf{d}} \right) d^2\mathbf{k}$.



Topological phase transition

- Assume $\mathcal{H}(\mathbf{k}; \alpha)$ and $|\psi_n(\mathbf{k}; \alpha)\rangle$ depends on parameter α . The Chern numbers of each band may be different for different α .



- A **topological phase** is a parameter region where all bands' topological invariance(s) keep constant.
- When parameter changes cross different phases, the system will undergo a **topological phase transition**, usually through band touching.

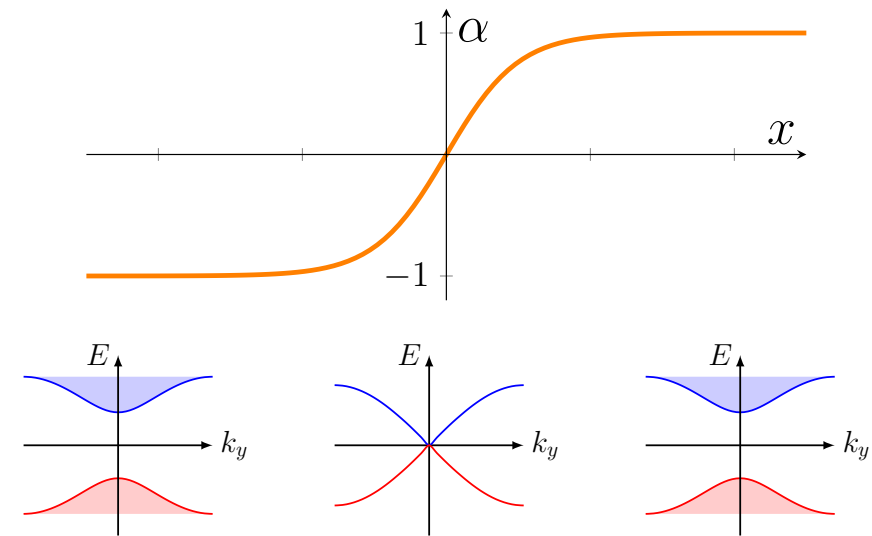
Topological edge modes

- If parameter α changes in the space, the location where the topological phase changes will have a gapless edge mode.

The bulk-edge correspondence

When two materials with differing topological phases and a common gapped spectrum are brought next to each other, modes localized to the interface and crossing the gap must appear at the interface.

- Physical intuition: topologically different bands can not continuously change to each other without band touching.
- Mathematically rigorous statement is possible by using the index theorem.



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Topological plasma waves

- Topological edge modes have been discovered in various plasma models:
 - Cold magnetized plasma^[1]
 - Ideal MHD plasma with constant magnetic shear^[2]
- Recently, our study^[3] has mapped out the topological phase diagram in cold magnetized plasmas, and thus gives the condition when topological edge modes exist.
- We also show that such edge modes can exist in a simple plasma-plasma interface.

Waves in cold magnetized plasma

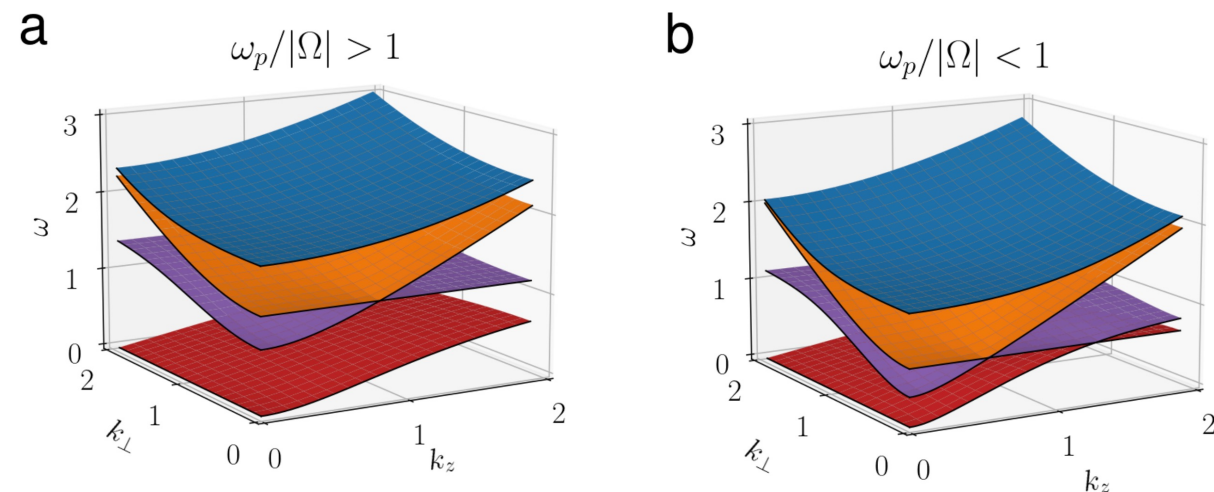
- We consider the electromagnetic waves in cold electron-ion plasma in 3D. Without base flow, $\mathbf{B}_0 = B_0 \hat{z}$, the linearized equations are:

$$\begin{aligned} \partial_t \mathbf{v}_1 &= \frac{e}{m_e} (\mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_0) \\ \partial_t \mathbf{E}_1 &= c^2 \nabla \times \mathbf{B}_1 - \frac{en_e}{\epsilon_0} \mathbf{v}_1 \\ \partial_t \mathbf{B}_1 &= -\nabla \times \mathbf{E}_1 \end{aligned} \quad \Rightarrow \quad i \frac{\partial}{\partial t} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{E}_1 \\ \mathbf{B}_1 \end{bmatrix} = \begin{bmatrix} i\Omega \hat{z} \times & i\omega_p & 0 \\ -i\omega_p & 0 & i\nabla \times \\ 0 & -i\nabla \times & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{E}_1 \\ \mathbf{B}_1 \end{bmatrix}$$

- Fourier transform gives: $\partial_t \rightarrow -i\omega$, $\nabla \rightarrow i\mathbf{k}$.
The Hamiltonian is $\mathcal{H}(\mathbf{k}; \omega_p, \Omega)$.
The dispersion relations $\omega(k_\perp, k_z)$ are:

- The bands touch each other only when:

$$|k_z| = k^\pm := \frac{\omega_p/c}{\sqrt{1 \pm \omega_p/\Omega}}$$



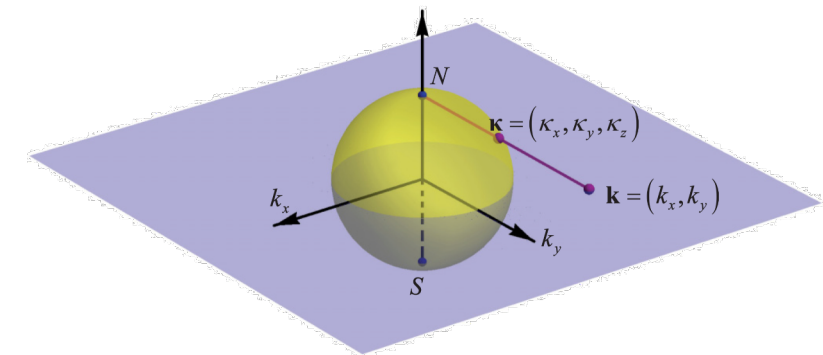
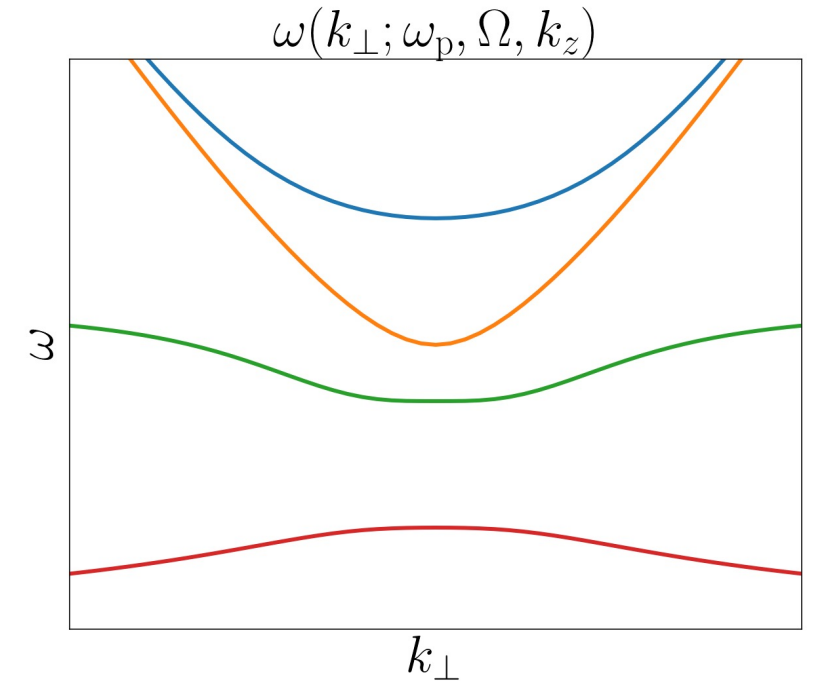
Reduce to 2D

- To characterize each band using Chern number, we further fixed k_z as a parameter.
- Notice here $(k_x, k_y) \in \mathbb{R}^2$, which is not a compact space. The Chern number may not be integers. However, the space can be compactified if:

$$k_{\perp} \rightarrow \infty, |\psi_n(k_x, k_y)\rangle \rightarrow |\tilde{\psi}_n\rangle.$$

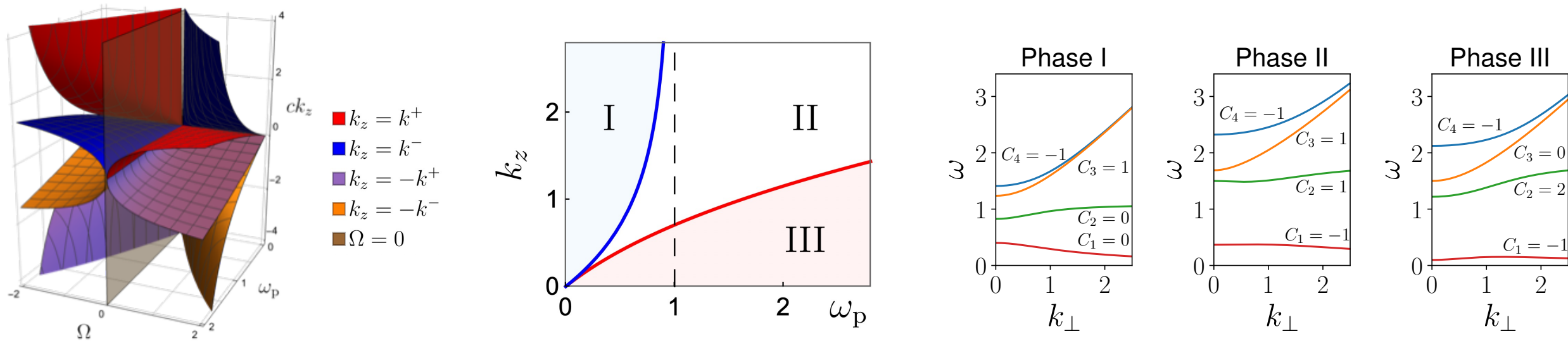
It is equivalent to compactify \mathbb{R}^2 into S^2 .

- If $|\psi_n\rangle$ does not satisfy the condition, a cut-off can be added artificially^[1]. This method is valid if the $k_{\perp} \rightarrow \infty$ behavior can be ignored in the system.



Topological phase diagram

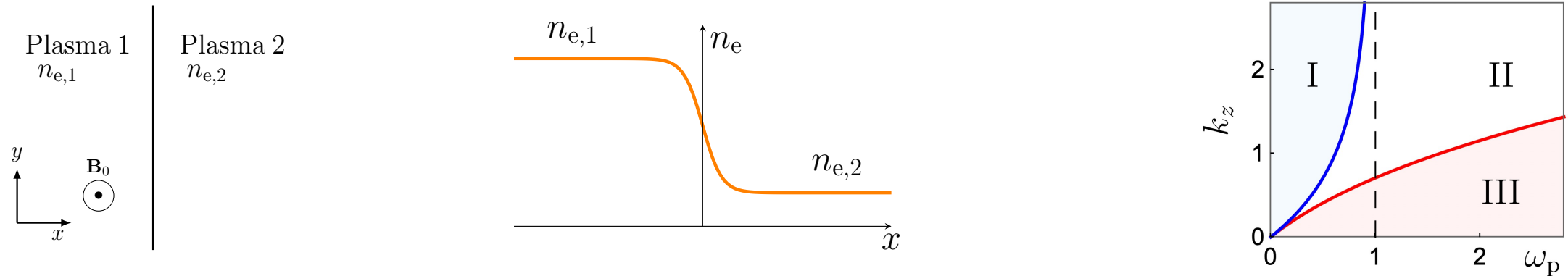
- At the band-touch points, Chern numbers may change. Therefore, the locations of k^\pm give the boundary of different (10) topological phases in the parameter space (ω_p, Ω, k_z) .



- For example, when $\Omega = 1$, the phase diagram (at $k_z > 0$) and Chern numbers shown above.
- Notice that phase I includes the case $\omega_p \rightarrow 0$, which is the vacuum.

Edge modes in non-uniform plasma

- Consider the following non-uniform plasma:



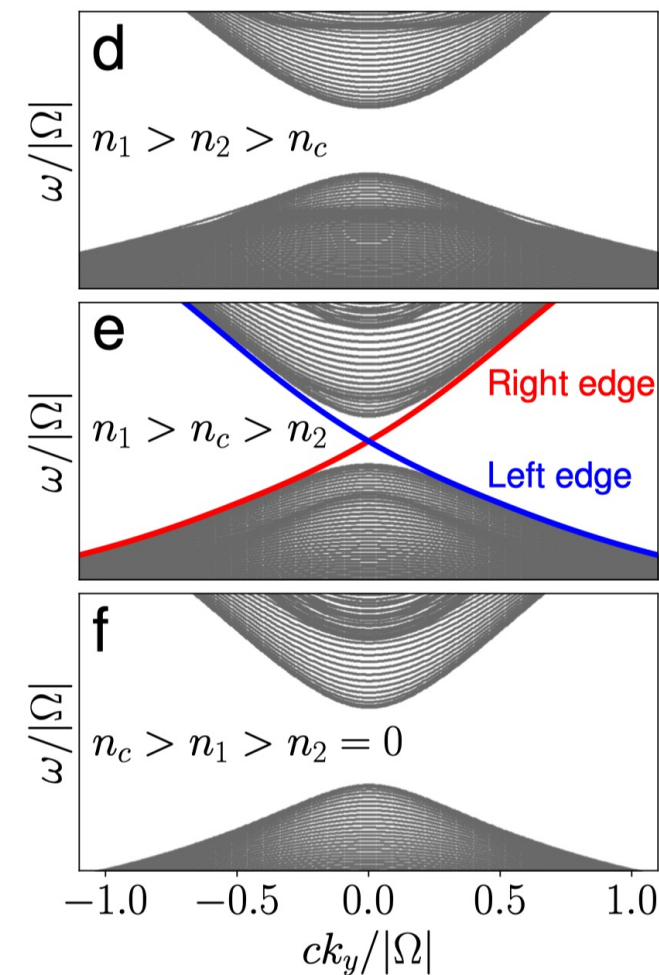
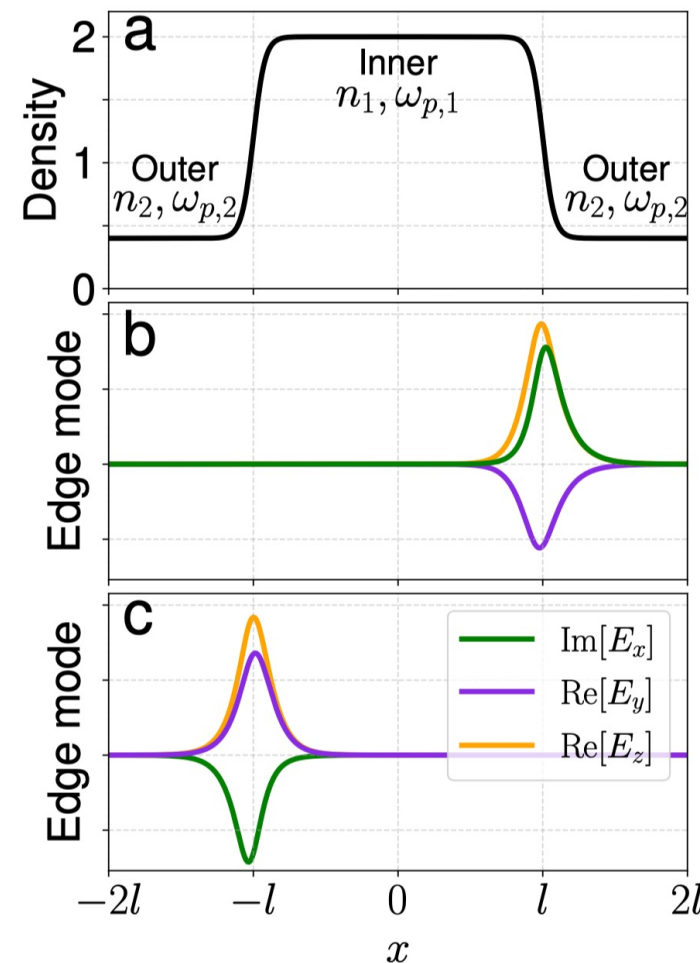
- For each given Ω, k_z . If and only if two sides of plasma are in different topological phases can the edge modes exist. The boundary of different phases gives a critical density n_c . The condition of the existence of edge modes is:

$$n_1 > n_c > n_2 \quad \Rightarrow \quad \frac{\omega_{p,1}}{|\Omega|} + \frac{\omega_{p,1}^2}{c^2 k_z^2} > 1 > \frac{\omega_{p,2}}{|\Omega|} + \frac{\omega_{p,2}^2}{c^2 k_z^2}$$

- The edge modes can exist not only in plasma-vacuum interface, but also inside the plasma at the interface between different density regions.

Numerical calculation of edge modes

- For convenience, in x-direction the density profile is chosen to be periodic.
- Edge modes only exist when the previous condition is satisfied.
- This type of edge mode localized at density gradient is a unique feature of gaseous systems.



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Conclusions

- A Hamiltonian $\mathcal{H}(\mathbf{k})$ maps a wave vector \mathbf{k} to its eigenvectors $|\psi_n(\mathbf{k})\rangle$. These maps define the topological phase of the system, which can be classified by various topological invariants.
- When two materials with different topological phases are put together, the topological edge modes may be found at the interface.
- In the cold magnetized plasma, there exist 10 topological phases in the parameter space (ω_p, Ω, k_z) . Topological edge modes can be found if plasmas with different topological phases are physically put together.
- These topological methods provide a new perspective to understand various waves in uniform and nonuniform plasmas.

Discussions and future researches

- The topological methods in fluids and plasmas were only developed recently. Some fundamental problems remain unsolved.
- Different plasma models usually have different topologies. The topology of more complicated and realistic plasma models is still unknown.
- In condensed matter physics, the existence of edge modes (quantum Hall effect) is closely related to Berry phase effect (Hall effect) of wave packets. Similar relation in continuous systems is unclear.
- Extending the topological ideas to non-Hermitian or non-linear systems is possible. However, the topologies are much more complicated and less understood.

Appendix: Chern numbers as Berry phases

- Assume we Hamiltonian $\mathcal{H}(\mathbf{R})$ and eigenvectors $|\psi_n(\mathbf{R})\rangle$. When $d\mathbf{R}(t)/dt \rightarrow 0$, the initial state $|\Psi_n(0)\rangle = |\psi_n(\mathbf{R}(0))\rangle$ will obtain a geometric phase $e^{i\gamma_n(t)}$:

$$\gamma_n(t) = i \int_0^t dt' \langle \psi_n(\mathbf{R}(t')) | \frac{d}{dt'} | \psi_n(\mathbf{R}(t')) \rangle = i \int_{\mathbf{R}(0)}^{\mathbf{R}(t)} d\mathbf{R} \langle \psi_n(\mathbf{R}) | \nabla_{\mathbf{R}} | \psi_n(\mathbf{R}) \rangle.$$

- Define Berry connection $\mathcal{A}_n(\mathbf{R}) = \langle \psi_n(\mathbf{R}) | \nabla_{\mathbf{R}} | \psi_n(\mathbf{R}) \rangle$, if $\mathbf{R}(t)$ goes around a closed loop, this phase becomes:

$$\gamma_n = \oint_{\partial\Sigma} \mathcal{A}_n(\mathbf{R}) \cdot d\mathbf{R} = \int_{\Sigma} (\nabla_{\mathbf{R}} \times \mathcal{A}_n) \cdot d\mathbf{S} = \int_{\Sigma} \mathbf{\Omega}_n \cdot d\mathbf{S}.$$

- We can choose \mathbf{R} to be the wave vector \mathbf{k} , then the Chern numbers becomes the total Berry phase in \mathbf{k} -space.

$$\frac{d\mathbf{p}}{dt} = \frac{d(\hbar\mathbf{k})}{dt} = q\mathbf{E} \quad \Rightarrow \quad \mathbf{k} = \mathbf{k}(t).$$

